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Elliptic Quadratic Forms, Focal Points, and A Generalized Theory of Oscillation

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The purpose of this paper is to generalize the theory, methods, and results for oscillation of second-order normal ordinary differential equations. This purpose is obtained by use of a theory of quadratic forms on Hilbert spaces given by Hestenes and the author.

In particular, the ideas of this paper may be applied to second-order abnormal problems of differential equations, higher-order control problems, integral and partial differential equations, abstract approximation problems, and to finite dimensional approximations which lead to meaningful computer algorithms.

For expository purposes some examples are included. Finally we show that specific existence and comparison theorems for the second-order case may be generalized to the $2n$th-order case.

1. INTRODUCTION

It appears that current methods for second-order normal oscillation problems cannot be easily extended to more general oscillation problems. For example, they will not easily extend to second-order abnormal problems of differential equations, to higher-order control problems, to integral and partial differential equations, nor to discrete approximation problems, where the discrete approximations may not vanish but should be called oscillatory. A further weakness of current methods is that (even for simple problems) they are qualitative and not quantitative.

The primary purpose of this paper is to consider a theory of oscillation which corrects the situations described above and which contains the second-order normal theory of oscillation. Our results will be based upon the quadratic form theory of Hestenes and on the focal point (and focal interval) theory of Hestenes and the author. In fact it seems that oscillation phenomena might more properly be called "focalization." We will use the former term for "commercial" purposes.

A major advantage of our unifying methods is that the groundwork for many difficult problems has been done. For example, Mikami has given a general theory for quadratic control problems; Lopez has given a general theory for (quadratic) higher-order derivatives including integral equations; and the author has given a general theory for approximation, including discrete and "computer type" results. A second advantage is that we obtain the "right" definition of oscillation for quadratic extremal problems and eigenvalue problems.

This paper is divided in the following way. Section 2 contains the necessary result from quadratic forms and focal point theory for the remainder of the paper. Since these ideas may appear obtuse, Sections 3 and 4 contain specific examples. The former section contains an elementary normal second-order problem; the latter section has a more complicated eight-order problem which illustrates many pertinent ideas of focal intervals. Section 5 gives the theory and some results for problems in the control setting. Section 6 gives some specific existence and comparison theorems for fourth-order equations in such a way that the generalization to the $2n$th-ordered case may be carried out as an exercise. Section 7 gives the theory of oscillation in the "calculus of variations" setting as opposed to the control setting of Section 5.

2. Preliminaries

In this section we give the necessary preliminary results concerning quadratic form theory and focal point theory. The former is contained in Refs. [3 or 6] while the latter is contained in Ref. [3].

Let $J(x)$ be a (real) elliptic quadratic form defined on a (real) Hilbert space $\mathscr{A}$ in the sense of Hestenes [6]. Let $\mathscr{B}^J$ denote the set of vectors J orthogonal to $\mathscr{B}$. The set $\mathscr{B}_0 = \mathscr{B} \cap \mathscr{B}^J$ will be called the J null vectors of $J(x)$ on $\mathscr{B}$. The dimension of $\mathscr{B}_0$ will be called the *nullity* of $J(x)$ on $\mathscr{B}$ and denote by $n(\mathscr{B})$. The *signature* (index) of $J(x)$ on $\mathscr{B}$, denoted by $s(\mathscr{B})$, is the dimension of a maximal $\mathscr{C}$ of $\mathscr{B}$ such that $y \neq 0$ in $\mathscr{C}$ implies $J(y) < 0$.

The indices $s(\mathscr{B})$ and $n(\mathscr{B})$ are finite and may be characterized in many ways. For example, the signature is the dimension of a maximal subspace $\mathscr{C}$ of $\mathscr{B}$ such that $J(x) \leqslant 0$ on $\mathscr{C}$ and $\mathscr{C} \cap \mathscr{B}_0 = 0$. The sum $s(\mathscr{B}) + n(\mathscr{B})$ is the dimension of a maximal subspace $\mathscr{D}$ of $\mathscr{B}$ such that $J(x) \leqslant 0$ on $\mathscr{D}$.

Let $\Lambda = [a, b]$ and assume for each λ in Λ there exists a closed subspace $\mathscr{B}(\lambda)$ of $\mathscr{B}$ such that $\mathscr{B}(a) = 0$, $\mathscr{B}(b) = \mathscr{B}$; $\lambda_1 < \lambda_2$ implies $\mathscr{B}(\lambda_1) \subset \mathscr{B}(\lambda_2)$;

$$\mathscr{B}(\lambda_0) = \bigcap_{\lambda_0 < \lambda \leqslant b} \mathscr{B}(\lambda) \qquad \text{whenever} \qquad a \leqslant \lambda_0 < b, \tag{1a}$$

$$\mathscr{B}(\lambda_0) = \left\langle \bigcup_{a \leqslant \lambda < \lambda_0} \mathscr{B}(\lambda) \right\rangle \qquad \text{whenever} \qquad a < \lambda \leqslant b. \tag{1b}$$

Let $s(\lambda)$ and $n(\lambda)$, respectively, denote the signature and nullity of $J(x)$ on $\mathscr{B}(\lambda)$. The symbol $s(\lambda + 0)$ will be the right-hand limit of $s(\lambda)$; $s(\lambda - 0)$, $n(\lambda + 0)$, $n(\lambda - 0)$ are analogously defined. We define $s(a - 0) = s(a) = 0$, $n(a - 0) = n(a) = 0$, $s(b + 0) = s(b)$, and $n(b + 0) = n(b)$. The real number λ in Λ is a *focal point* if $s(\lambda + 0) \neq s(\lambda - 0)$. The difference $f(\lambda) = s(\lambda + 0) - s(\lambda - 0)$ will be called *the order* of λ as a focal point.

The following results have been given in Ref. [3], Theorem 1 allows for the existence of nondegenerate focal intervals (corresponding to abnormal solutions) while Theorem 2 corresponds to the more usual normal case. Focal intervals have been characterized in Reference [3].

THEOREM 1. *Let λ_0 be given such that $a \leqslant \lambda_0 < b$. Then the following inequalities hold:*

$$s(\lambda_0 - 0) = s(\lambda_0), \qquad s(\lambda_0 + 0) \geqslant s(\lambda_0) \tag{2a}$$

$$n(\lambda_0) \geqslant n(\lambda_0 - 0), \qquad n(\lambda_0) \geqslant n(\lambda_0 + 0) \tag{2b}$$

$$s(\lambda_0 + 0) - s(\lambda_0) = n(\lambda_0) - n(\lambda_0 + 0) \geqslant 0. \tag{2c}$$

THEOREM 2. *If $\mathscr{B}_0(\lambda_1) \cap \mathscr{B}_0(\lambda_2) = 0$ when $\lambda_1 \neq \lambda_2$, then $f(a) = 0$ and $f(\lambda) = n(\lambda)$ on $a \leqslant \lambda \leqslant b$. Thus if λ_0 in Λ the following quantities are equal:*

(3a) *the number of focal points on $a \leqslant \lambda < \lambda_0$,*

(3b) *the signature $s(\lambda_0)$ of $J(x)$ on $\mathscr{B}(\lambda_0)$,*

(3c) *the sum $\sum_{a\leqslant\lambda<\lambda_0} n(\lambda)$, and*

(3d) *the sum $\sum [s(\lambda_i + 0) - s(\lambda_i)]$ taken over all λ_i such that $a \leqslant \lambda_i < \lambda_0$ and $s(\lambda)$ is discontinuous at λ_i.*

3. AN ELEMENTARY EXAMPLE

To illustrate Theorem 2 we give an elementary example. Thus, let

$$J(x) = \int_0^\pi (\dot{x}^2 - 16x^2)\, dt$$

be defined relative to a one-parameter family of subspaces $\mathscr{B}(\lambda)$ $(0 \leqslant \lambda \leqslant \pi)$ of arcs $x(t)$, where $x(t)$ is absolutely continuous, $\dot{x}(t)$ (the derivate of $x(t)$) is square integrable, with boundary conditions

$$x(0) = 0 \qquad \text{and} \qquad x(t) = 0 \quad \text{on} \quad \lambda \leqslant t \leqslant \pi.$$

It is immediate that $x(t)$ is in $\mathscr{B}_0(\lambda)$ if and only if $x(t)$ satisfies the above boundary conditions and $\ddot{x} + 16x = 0$; that is, $x(t) = \sin 4t$ on $0 \leqslant t \leqslant \lambda$ and $x(t) = 0$ on $\lambda < t \leqslant \pi$.

The following results agree with Theorem 2:

$$n(\lambda) = \begin{cases} 1 & \text{if} \quad \lambda = \pi/4, \pi/2, 3\pi/4, \pi \\ 0 & \quad\quad \text{otherwise,} \end{cases}$$

and

$$s(\lambda) = \begin{cases} 0 & \text{if} \quad \lambda \leqslant \pi/4 \\ 1 & \text{if} \quad \pi/4 < \lambda \leqslant \pi/2 \\ 2 & \text{if} \quad \pi/2 < \lambda \leqslant 3(\pi/4) \\ 3 & \text{if} \quad 3(\pi/4) < \lambda \leqslant \pi. \end{cases}$$

We note that at the points $\lambda = \pi/4, \pi/2$, and $3\pi/4$ we have gained a negative vector. This suggests that the usual second-order normal oscillation problems can be studied as extremal problems. Furthermore (using Ref. [4]) these oscillation problems can be studied by approximation methods.

4. A Further Example

The purpose of this section is to give an example for Theorem 1. In this case we have abnormal solutions to our differential equations and hence nondegenerate focal intervals. It seems that the usual oscillation arguments will not carry over from second-order equations to this example. This example has been given in Ref. [9] with other purposes and methods in mind. Omitted details will be found in the next section or in Refs. [8 and 9].

Thus let

$$J(x) = \int_0^\pi (u^*u - x^*x)\, dt \tag{4}$$

be defined relative to a one-parameter family of subspaces $\mathscr{C}(\lambda)$ $(0 \leqslant \lambda \leqslant \pi)$ of arcs x: $x^i(t)$, $u^k(t)$ $(0 \leqslant t \leqslant \pi;\ i, k = 1,..., 4)$ such that

$$\dot{x} = Bu \qquad (0 \leqslant t \leqslant \lambda) \tag{5}$$

with boundary conditions

$$x(0) = 0 \tag{6a}$$

and

$$x(t) = 0, \qquad u(t) = 0 \qquad (\lambda \leqslant t \leqslant \pi). \tag{6b}$$

The 4×4 matrix $B(t)$ has been chosen to be

$$\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

respectively on the intervals $0 \leqslant t < \pi/2$, $\pi/2 \leqslant t < 3(\pi/4)$, and $3(\pi/4) < t \leqslant \pi$. In the above v^* and $\dot{v}$, respectively, denote the transpose and derivate of v.

In the next section we show that the vector $x(t)$ is in $\mathscr{C}_0(\lambda)$ if and only if there exists an absolutely continuous vector $p: p^i(t)$ $(0 \leqslant t \leqslant \pi;\ i = 1,..., 4)$ such that (5), (6), and

$$\dot{p} = -x \qquad (0 \leqslant t \leqslant \lambda), \tag{7a}$$

$$B^*p = u \qquad (0 \leqslant t \leqslant \lambda) \tag{7b}$$

hold. We note that (5) and (7a) become

$$\dot{x} = BB^*p, \qquad \dot{p} = -x. \tag{8}$$

There exist four linearly independent solutions (x, p) of (8) satisfying (6a). Denoting these solutions by (X_a, P_a), (X_b, P_b), (X_c, P_c), and (X_d, P_d) the reader may verify that the solutions are given by

$$\begin{aligned}
X_a^1 &= 2 \sin 2t, 0, -2 \sin(2t - 3\pi/2) \\
P_a^1 &= \cos 2t, -1, -\cos(2t - 3\pi/2) \\
X_a^2 &= 2 \sin 2t, 0, 0 \\
P_a^2 &= \cos 2t, -1, -1 \\
X_b^1 &= 0, 0, -2 \sin(2t - 3\pi/2) \\
P_b^1 &= -1, -1, -\cos(2t - 3\pi/2) \\
P_b^2 &= 1, 1, 1 \\
X_c^3 &= 2 \sin 4t, 0, -\sin(2t - 3\pi/2) \\
P_c^3 &= \tfrac{1}{2} \cos 4t, \tfrac{1}{2}, \tfrac{1}{2} \cos(2t - 3\pi/2) \\
X_d^4 &= 2 \sin t, 2 \sin t, 2 \sin t \\
P_d^4 &= 2 \cos t, 2 \cos t, 2 \cos t,
\end{aligned}$$

where the "commas" denote the intervals $0 \leqslant t < \pi/2$, $\pi/2 \leqslant t < 3\pi/4$, and $3\pi/4 \leqslant t \leqslant \pi$, respectively; and omitted solutions are identically zero.

We note that there is a solution to (8) which vanishes at each point of $0 \leqslant t \leqslant \pi$, since $X_b(t) = 0$ on $0 \leqslant t \leqslant 3\pi/4$ while $X_a(t) - X_b(t) = 0$ on

$3\pi/4 \leqslant t \leqslant \pi$. Thus we expect that the usual oscillation methods will not be effective.

From Ref. [3], we note that λ is a focal point if and only if λ is the right-hand end point of a focal interval which cannot be extended in either direction and whose left endpoint is not $t = 0$. Thus $\lambda = \pi/4$ is a focal point of order 1, since it is a (degenerate) focal interval for X_c. In addition $\lambda = 3\pi/4$ is a focal point of order 1 since the focal interval $I = \{t: \pi/2 \leqslant t \leqslant 3\pi/4\}$ associated with X_c is not to be extended in either direction. We note that $\lambda = 3\pi/4$ does not have order 2 since the solution $X_a - X_b$ is zero to the right of $\lambda = 3\pi/4$. Finally, if $B(t)$ is extended smoothly at $t = \pi$, then $\lambda = \pi$ would be a focal point associated with X_d and possibly with $X_a - X_b$.

In terms of the indices $s(\lambda)$ and $n(\lambda)$ we have

$$s(\lambda) = \begin{cases} 0 & \text{if} \quad 0 \leqslant \lambda \leqslant \pi/4 \\ 1 & \text{if} \quad \pi/4 < \lambda \leqslant 3\pi/4 \\ 2 & \text{if} \quad 3\pi/4 < \lambda \leqslant \pi \end{cases}$$

and

$$n(\lambda) = \begin{cases} 0 & \text{if} \quad 0 \leqslant \lambda < \pi/4 \quad \text{or} \quad \pi/4 < \lambda < \pi/2 \\ 1 & \text{if} \quad \lambda = \pi/4 \quad \text{or} \quad 3\pi/4 < \lambda < \pi \\ 2 & \text{if} \quad \pi/2 \leqslant \lambda \leqslant 3\pi/4 \quad \text{or} \quad \lambda = \pi. \end{cases}$$

5. A New Definition of Oscillation for Quadratic Control Problems

In this section we give a new definition of oscillation. It is immediate that "oscillation points" are really focal points. For commercial reasons we have chosen the former wording. It is shown that this definition reduces to the usual definition for the usual (normal second-order) problems related to Theorem 2. It appears that this definition is the natural one to use for higher-order quadratic control and integral equation problems, approximation results, quantitative results, and more abstract systems. Thus let

$$J(x) = \int_a^b 2\omega(t, x, u)\, dt, \tag{9a}$$

where

$$2\omega(t, x, u) = x^*P(t)\,x + x^*Q(t)\,u + u^*Q^*(t)\,x + u^*R(t)\,u \tag{9b}$$

is given relative to a one-parameter family $\mathscr{B}(\lambda)$ $(a \leqslant \lambda \leqslant b)$ of arcs (x, u) satisfying the control equation

$$\dot{x} = Ax + Bu \qquad (a \leqslant t \leqslant \lambda), \tag{10a}$$

a linear constraint

$$Mx + Nu = 0 \qquad (a \leqslant t \leqslant \lambda), \tag{10b}$$

and the boundary conditions

$$x(a) = 0 \tag{11a}$$

$$x(t) = 0, \qquad u(t) = 0 \qquad (\lambda \leqslant t \leqslant b). \tag{11b}$$

In the above, $x(t)$ is absolutely continuous; $\dot{x}(t)$ and $u(t)$ are square integrable; A, $P = P^*$, and Q are square integrable; B, M, N, and $R = R^*$ are essentially bounded and measurable. The vectors x and u, respectively, are n and q dimensional while * denotes matrix transpose. In addition we assume $J(x)$ is elliptic in the sense of Hestenes (see Ref. [8]). The following theorem is given in Ref. [8 or 9].

THEOREM 3. *The vector x is J orthogonal to $\mathscr{B}(\lambda)$ if and only if there exists an absolutely continuous vector $p(t)$ $(a \leqslant t \leqslant \lambda)$ and a square integrable vector $v(t)$ $(a \leqslant t \leqslant \lambda)$ such that*

$$\dot{p} + \overset{*}{A}p + M^*v = \omega_x \tag{12a}$$

and

$$\overset{*}{B}p + \overset{*}{N}v = \omega_u \tag{12b}$$

hold on $a \leqslant t \leqslant \lambda$. Thus x is in $\mathscr{B}_0(\lambda)$ if and only if (10) *and* (11) *hold and* (12) *holds on $a \leqslant t \leqslant \lambda$.*

The control problem given by (9), (10), and (11) is *oscillatory of degree m if $s(b) = m$*. Equation (12) is oscillatory of degree m on $a \leqslant t \leqslant b$ if there exists vectors (x, u, p, v) satisfying the above continuity conditions, such that (x, u) is J orthogonal $\mathscr{B}(b)$ and the well-defined associated control problem is oscillatory of degree m. In this case the vector (x, u) is an oscillatory solution of degree m on $a \leqslant t \leqslant b$. Equation (12) is oscillatory if for any integer $M > 0$ there exists real numbers $a < b$ such that Eq. (12) is oscillatory of degree m on $a \leqslant t \leqslant b$ and $m \geqslant M$. Finally, Eq. (12) is nonoscillatory if it is not oscillatory.

For second-order differential equations, the usual definition of oscillatory is that a nonzero solution of the associated differential equation have arbitrary large zeros on $[0, \infty)$. Since these problems are normal in that they correspond to Theorem 2, our definition is more general. In this case (12) becomes

$$(rx')' - px = 0, \tag{12$'$}$$

with $r(t) > 0$. Thus we have the following.

THEOREM 4. *If a nonzero solution of Eq.* (12′) *has arbitrary large zeros on* $[0, \infty)$ *and if Equation* (12′) *is normal then it is oscillatory* (in our sense).

Let $x(t)$ be the solution described above such that $x(\lambda) = 0$. Let $y(t) = x(t)$ on $[0, \lambda]$, and set $y(t) \equiv 0$ otherwise. By Theorem 3, $y(t)$ is in $\mathscr{B}_0(\lambda)$, and thus λ is a focal point by Theorem 2. The result now follows from the definition of oscillation.

We note that Theorem 4 may be extended to the abnormal situation. In this case we mean by "arbitrary large zeros on $[0, \infty)$" the focal interval phenomena of Section 4.

We will now consider comparison theorems for more general oscillation problems. For this purpose (for $i = 1, 2$) we set

$$2\omega_i(t, x, u) = \mathring{x} P_i(t)\, x + u^* R_i(t)\, u, \tag{9b$_i$}$$

with corresponding changes to Eqs. (9a), (10), and (12) and to A_i, B_i, M_i and N_i in Eqs. (10_i) and (12_i). We assume both problems are elliptic in the sense of Hestenes. In Theorem 5, $T_2 \geqslant T_1$ means $T = T_2 - T_1$ is a nonnegative definite matrix.

THEOREM 5. *If* $R_1(t) \geqslant R_2(t)$, $P_2(t) \leqslant P_1(t)$ *and Eq.* (12_1) *is oscillatory of degree* M_1, *then Eq.* (12_2) *is oscillatory of degree* $M_2 \geqslant M_1$. *Thus if* (12_1) *is oscillatory then* (12_2) *is oscillatory; if* (12_2) *is nonoscillatory then* (12_1) *is nonoscillatory.*

This theorem follows immediately from Theorem 3 and the fact that if (x, u) is such that $J_1(x) < 0$ then $J_2(x) \leqslant J_1(x) < 0$.

More generally we have the following.

THEOREM 6. *If Eq.* (12_1) *is oscillatory and the focal points of the* $i = 2$ *control system interlace the focal points of the* $i = 1$ *system, then Eq.* (12_2) *is oscillatory.*

If we assume the normal setting we have by Theorem 2 and 3 the more familiar Theorem 7.

THEOREM 7. *Assume Eq.* (12_1) *is oscillatory and between any two consecutive zeros* $t_1 < t_2$ *of a nonzero solution* $x(t)$ *of* (12_1) *there exists a nonzero vector* $y(t)$ *vanishing outside* (t_1, t_2) *such that* $J_2(y) \leqslant 0$. *Then Eq.* (12_2) *is oscillatory.*

We note that proper interpretation of Theorem 7 leads to many oscillation results (for example, Theorems 10 and 12).

6. The Fourth-Order Normal Problem

In this section we will consider the normal fourth-order case. The extension of these results to the general (abnormal), $2n$th-order case ($n \geqslant 3$) is immediate and left as an exercise. Thus let

$$J(z) = \int_a^b [r(t)\, \ddot{z}^2 - q(t)\, z^2]\, dt \tag{13}$$

be given relative to a one parameter family $\mathscr{C}(\lambda)$ ($a \leqslant \lambda \leqslant b$) of arcs $x = (z, \dot{z})^*$ such that $z(t)$ is continuously differentiable, $\dot{z}$ is absolutely continuous, and $\ddot{z}$ is square integrable on $a \leqslant t \leqslant \lambda$ satisfying

$$x(a) = [z(a), \dot{z}(a)]^* = 0 \tag{14a}$$

$$x(t) = [z(t), \dot{z}(t)]^* = 0 \qquad (\lambda \leqslant t \leqslant b). \tag{14b}$$

The above is the normal fourth-order problem with

$$R(t) = \begin{pmatrix} 0 & 0 \\ 0 & r(t) \end{pmatrix}, \qquad P(t) = \begin{pmatrix} -q(t) & 0 \\ 0 & 0 \end{pmatrix}$$

in Eq. (9b);

$$A(t) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in Eq. (10a); and

$$M(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad N(t) = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$$

in Eq. (10b).

The conditions of ellipticity are satisfied, since the strengthened Clebsch conditions (see Ref. [8]) hold if $r(t) \geqslant \epsilon > 0$. For in this case

$$\pi^* R(t)\, \pi + h_1 \pi^* N^* N \pi \geqslant h_0 \pi^* \pi \qquad (\text{for } \pi \text{ in } E^2)$$

holds for almost all t in $a \leqslant t \leqslant b$ with $h_1 = h_0 = \epsilon$. Letting $z'(t)$ denote the derivative of z (when convenient); Eqs. (12a) and (12b), respectively, become

$$\begin{pmatrix} p_1' \\ p_2' \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = \omega_x = \begin{pmatrix} -q & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} z \\ z' \end{pmatrix} \tag{15a}$$

and

$$\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} = R(t)\, x' = \begin{pmatrix} 0 & 0 \\ 0 & r(t) \end{pmatrix} \begin{pmatrix} z' \\ z'' \end{pmatrix}, \tag{15b}$$

which becomes

$$(rz'')'' - qz = 0. \tag{16}$$

THEOREM 8. *The vector $z(t)$ is J orthogonal to $\mathscr{C}(\lambda)$ if and only if there exists an absolutely continuous vector $p(t) = [p_1(t), p_2(t)]^*$ $(a \leqslant t \leqslant \lambda)$ and a square integrable vector $\mu(t) = [\mu_1(t), \mu_2(t)]^*$ $(a \leqslant t \leqslant \lambda)$ such that Eqs. (15) hold on $a \leqslant t \leqslant \lambda$. Thus z is in $\mathscr{C}_0(\lambda)$ if and only if (15) holds on $a \leqslant t \leqslant \lambda$; $z(a) = z'(a) = 0$; $z(t) = z'(t) = 0$ on $\lambda \leqslant t \leqslant b$.*

COROLLARY 1. *Let $a < \lambda < b$. Then $n(\lambda) \neq 0$, or equivalently, λ is a focal point, if and only if there exists $z(t) \neq 0$ satisfying (16) on $a \leqslant t \leqslant b$ such that $z(a) = \dot{z}(a) = 0$ and $z(\lambda) = \dot{z}(\lambda) = 0$.*

Letting $y = (y_1, y_2, y_3, y_4)^*$ where $y_1 = z, y_2 = z', y_3 = rz'', y_4 = (rz'')'$, Eq. (16) becomes the more familiar linear system $S(t)y' + T(t)y = 0$ or $y' = U(t)y$ where

$$S(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T(t) = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -q & 0 & 0 & 0 \end{pmatrix},$$

and

$$U(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/r & 0 \\ 0 & 0 & 0 & 1 \\ q & 0 & 0 & 0 \end{pmatrix}.$$

For illustration purposes we present a constructive type which is not "best possible" but illustrates some important concepts.

THEOREM 9. *Assume $r(t)$ and $q(t)$ in Eq. (16) satisfy $0 < h \leqslant r(t) \leqslant \epsilon_1$ and $q(t) \geqslant \epsilon_2 > 0$ on an interval $t_1 \leqslant t \leqslant t_2$ of length*

$$t_2 - t_1 = M \geqslant 4\left(\frac{8\epsilon_1}{3\epsilon_2}\right)^{1/4}.$$

Then there exists a nontrivial solution of (16) which satisfies

$$z(\xi_i) = z'(\xi_i) = 0 \quad (i = 1, 2) \qquad \textit{for } t_1 \leqslant \xi_1 < \xi_2 < t_2 .$$

The ideas are "variations on a theme" that the author has used for second-order equations.

Thus let

$$y(t) = \begin{cases} (t - t_1)^2 & \text{if} \quad t_1 \leqslant t \leqslant \dfrac{3t_1 + t_2}{4} \\ -\left(t - \dfrac{t_1 + t_2}{2}\right)^2 + 2\left(\dfrac{t_2 - t_1}{4}\right)^2 & \text{if} \quad \dfrac{3t_1 + t_2}{4} < t \leqslant \dfrac{t_1 + 3t_2}{4} \\ (t - t_2)^2 & \text{if} \quad \dfrac{t_1 + 3t_2}{4} < t \leqslant t_2 \\ 0 & \text{otherwise} \end{cases}$$

Letting $\eta = (t_2 - t_1)/4$ we have

$$\int_{t_1}^{t_2} (ry''^2 - qy^2)\, dt \leqslant 4\epsilon_1 M - \epsilon_2 I \leqslant 4\epsilon_1 M - 6\epsilon_2 \left(\frac{M}{4}\right)^5 \leqslant 0$$

since

$$\begin{aligned} I = \int_{t_1}^{t_2} y^2(t)\, dt &= \int_0^\eta s^4\, ds + \int_{-\eta}^{\eta} [-s^2 + 2\eta^2]\, ds - \int_\eta^0 s^4\, ds \\ &\geqslant 2\int_0^\eta s^4\, ds + 2\int_0^\eta [s^4 - 4s^2\eta^2 + 4\eta^4]\, ds \\ &= \eta^5[8 - 8/3 + 4/5] \geqslant 6\eta^5. \end{aligned}$$

Thus $s(t_2) \geqslant s(t_1) + 1$ and our result follows by Theorems 2 and 8.

It is interesting to note that the above procedure can be extended to more general (abnormal) cases including the normal equation

$$(rz^{(n)})^{(n)} - qz = 0. \tag{17}$$

The rationale is to choose $y(t)$ vanishing outside of (t_1, t_2) with the "control" $y^{(n)}(t) = 1$ on the interval $[t_1, t_1 + \eta)$, $y^{(n)}(t) = -1$ on $(t_1 + \eta, t_1 + 3\eta)$,..., $y^{(n)}(t) = -1$ on $(t_2 - 3\eta, t_2 - \eta)$, and $y^{(n)}(t) = 1$ on $(t_2 - \eta, t_2)$, where $\eta = (t_2 - t_1)/2^n$. For each n we have constructed a "negative" vector, so that $s(t_2) \geqslant s(t_1) + 1$ if $t_2 - t_1 = M \geqslant k(n)\,(\epsilon_1/\epsilon_2)^{1/2n}$ where $k(n)$ depends on n.

We now give two sufficiency comparison theorems for fourth-order oscillation problems. As might be expected the results are weaker than the similar result for second-order equations given by the author. Thus let

$$[r_1(t)\, x'']'' - c^2(t)\, q_1(t)\, x = 0 \tag{18}$$

be associated with a second form

$$J_1(x) = \int_a^b [r_1(t)\, x''^2 - c^2(t)\, q_1(t)\, x^2]\, dt, \tag{19}$$

which is given relative to the spaces $\mathscr{C}(\lambda)$ defined above. To simplify the text we assume r, q, r_1, q_1 are continuous on $a \leqslant t \leqslant b$ while each condition $c^2(t) \geqslant 1$, $c'' = 0$, and $[r(c^2)']' \leqslant 0$ holds piecewise on $a \leqslant t \leqslant b$, so that the integrals given below exist. Finally, assume $r(t) \geqslant r_1(t) > 0$ and $q(t) \leqslant q_1(t)$ hold on $a \leqslant t \leqslant b$.

THEOREM 10. *If $x \neq 0$ satisfies* (16) *on* $[t_1, t_2]$ *and*

$$x(t_1) = x(t_2) = x'(t_1) = x'(t_2) = 0,$$

then there exists a nontrivial solution $w(t)$ of (18) *satisfying $w(\xi) = w'(\xi) = 0$ for some ξ in $[t_1, t_2)$. Thus if* (16) *is oscillatory so is* (18).

Let $x(t) = c(t)\, y(t)$ on $[t_1, t_2]$ and $y(t) = 0$ otherwise. Then

$$y(t_1) = y(t_2) = 0, \qquad y'(t_1) = y'(t_2) = 0$$

and

$$\begin{aligned} 0 &= \int_{t_1}^{t_2} (rx''^2 - qx^2)\, dt = \int_{t_1}^{t_2} \{r[cy'' + 2c'y' + c''y]^2 - qc^2y^2\}\, dt \\ &= \int_{t_1}^{t_2} [r_1 y''^2 - c^2 q_1 y^2]\, dt + \int_{t_1}^{t_2} \{(c^2 r - r_1)\, y''^2 - c^2(q - q_1)\, y^2\}\, dt \\ &\quad + 4\int_{t_1}^{t_2} r[(c'y')^2 + cc'y'y'']\, dt. \end{aligned} \tag{20}$$

Since

$$0 = \int_{t_1}^{t_2} \frac{d}{dt}\{[r(c^2)']\,(y')^2\}\, dt = \int_{t_1}^{t_2} \{[r(c^2)']'\,(y')^2 + 4rcc'y'y''\}\, dt,$$

the second and third integrals of the previous equality are nonnegative. Thus

$$\int_{t_1}^{t_2} [r_1 y''^2 - c^2 q_1 y^2]\, dt \leqslant 0$$

and $s(t_2) \geqslant s(t_1) + 1$. The result follows by Theorems 2 and 8.

More generally we set $\rho(t) = 2cc''$, $\epsilon(t) = 2cc' = (c^2)'$, $\eta(t) = c'c'' - cc'''$ and assume $c(t)$ satisfies $c^2(t) \geqslant 1$, $cc'''' \geqslant 0$, and $2cc'' \leqslant c'^2$ piecewise on $[t_1, t_2]$. Then

$$\int_{t_1}^{t_2} \frac{d}{dt}[\epsilon y'^2 + \rho(yy') + \eta y^2]\, dt = 0$$

whenever

$$y(t_1) = y(t_2) = y'(t_1) = y'(t_2) = 0$$

so that

$$-\int_{t_1}^{t_2} (c''y + 2c'y')(c''y + 2c'y' + 2cy'')\, dt$$

$$= \int_{t_1}^{t_2} \{(\eta' - c''^2)\, y^2 + (\rho' + 2\eta - 4c'c'')\, yy' + (\epsilon' + \rho - 4c'^2)\, y^2 \tag{21}$$

$$+ (2\epsilon - 4cc')\, y'y'' + (\rho - 2cc'')\, yy''\}\, dt \leqslant 0.$$

COROLLARY 2. *Assume $r(t)$, $r_1(t)$, $q(t)$ and $q_1(t)$ satisfy the conditions of Theorem* 10, *and $c(t)$ satisfies the above conditions. Then Theorem* 10 *holds.*

Replace the last integral in (20) by

$$\int_{t_1}^{t_2} \{r(c''y + 2c'y')(c''y + 2c'y' + 2cy'')\}\, dt.$$

This integral is nonnegative by inequality (21) and the meanvalue theorem. The result now follows as in Theorem 10.

We note that the process of (differentially) completing the square yields corresponding comparison results between Eq. (17) and

$$r_1 z^{(2n)} - c^2 q_1 z = 0. \tag{22}$$

7. OSCILLATION THEORY IN THE CALCULUS OF VARIATIONS SETTING

In this section we will consider higher-order quadratic problems in a calculus of variations setting as opposed to the control setting of Section 5. In many problems the two settings, when viewed properly, yield equivalent results. Even in this case it is usually more convenient to work in one setting as opposed to the other.

We note that the Euler–Lagrange equations described below may be found in Ref. [11]. This reference also contains the same background ideas necessary for the study of oscillation problems for integral equations. The latter topic will not be considered here.

The fundamental Hilbert space considered in this section is the set of real valued functions $z(t) = [z_1(t),..., z_p(t)]$, whose αth component, $z_\alpha(t)$ is a real-valued function defined on the interval $a \leqslant t \leqslant b$ of class C^{n-1}; $z_\alpha^{(n-1)}(t)$

is absolutely continuous and $z_\alpha^{(n)}(t)$ is Lebesque square integrable on $a \leqslant t \leqslant b$. The inner product is given by

$$(x, y) = x_\alpha^{(k)}(a)\, y_\alpha^{(k)}(a) + \int_a^b x_\alpha^{(n)}(t)\, y_\alpha^{(n)}(t)\, dt,$$

where we assume $\alpha = 1,..., p$; $k = 0,..., n-1$; the superscripts denote the order of differentiation and the above expression is summed with respect to α and k.

The fundamental quadratic form $J(x)$ is given by

$$J(x) = \int_a^b R_{\alpha\beta}^{ij}(t)\, x_\alpha^{(i)}(t)\, x_\beta^{(j)}(t)\, dt, \tag{23}$$

($\alpha, \beta = 1,..., p$; $i, j = 0,..., n$), where $R_{\alpha\beta}^{ij}(t) = R_{\beta\alpha}^{ji}(t)$ are essentially bounded and integrable functions on $a \leqslant t \leqslant b$, and the inequality

$$R_{\alpha\beta}^{nn}(t)\, \pi_\alpha \pi_\beta \geqslant h \pi_\alpha \pi_\beta \qquad (\alpha, \beta = 1,..., p;\ \alpha, \beta \text{ summed}) \tag{24}$$

holds almost everywhere on $a \leqslant t \leqslant b$ for every $\pi = (\pi_1 ,..., \pi_p)$ in E^p and some $h > 0$. Inequality (24) is the ellipticity condition for quadratic problems in this setting.

Let $\mathscr{B}$ denote the linear subspace whose component functions $x_\alpha(t)$ satisfy $x_\alpha{}^k(a) = x_\alpha{}^k(b) = 0$, and let $\mathscr{B}(\lambda)$ ($a \leqslant \lambda \leqslant b$) denote the subclass whose component functions satisfy (for $\alpha = 1,..., p$; $k = 0,..., n-1$)

$$x_\alpha^{(k)}(a) = 0 \tag{25a}$$

and

$$x_\alpha(t) \equiv 0 \qquad \text{on} \qquad \lambda \leqslant t \leqslant b. \tag{25b}$$

In this section we assume $\alpha = 1,..., p$ and $k = 0,..., n-1$. For the next theorem let $x = [x_1 ,..., x_p]$ and set

$$\tau_\beta{}^j(t) = R_{\alpha\beta}^{ij}(t)\, x_\alpha^{(i)}(t), \tag{26a}$$

$$v_\beta{}^0(t) = \int_a^t \tau_\beta{}^0(s)\, ds + c_\beta{}^0, \tag{26b}$$

and

$$v_\beta{}^\ell(t) = \int_a^t [\tau_\beta{}^\ell(s) - v_\beta^{\ell-1}(s)]\, ds + c_\beta{}^\ell, \tag{26c}$$

where (26a) holds almost everywhere on $a \leqslant t \leqslant b$; $\alpha, \beta = 1,..., p$; $i, j = 0,..., n$; $l = 1,..., n-1$; and $c_\beta{}^0,..., c_\beta^{n-1}$ denote ($n \times p$) real constants.

THEOREM 11. *The vector* $x = [x_1(t),..., x_p(t)]$ *is* J *orthogonal to* $\mathscr{B}(\lambda)$ *if and only if there exist constants* $c_\beta^0,..., c_\beta^{n-1}$ *in* (26) *such that the Euler equations given by*

$$\tau_\beta{}^n(t) = v_\beta^{n-1}(t) \tag{27}$$

hold almost everywhere on $a \leqslant t \leqslant \lambda$, *where* $\tau_\beta{}^n(t)$ *is given by* (26a) *and* $v_\beta^{n-1}(t)$ *is given by* (26c). *Thus* x *is in* $\mathscr{B}_0(\lambda)$ *if and only if Eq.* (25) *holds and* (27) *is satisfied almost everywhere on* $a \leqslant t \leqslant \lambda$.

For illustrative purposes we note that the problem considered in Section 6 becomes with $p = 1$, $n = 2$:

$$R^{ij}(t) = \begin{cases} r(t) & \text{if} \quad i = j = 2 \\ -p(t) & \text{if} \quad i = j = 0 \\ 0 & \qquad \text{otherwise,} \end{cases}$$

$$\tau^i(t) = \begin{cases} -p(t)\,x(t) & \text{if} \quad i = 0 \\ r(t)\,x''(t) & \text{if} \quad i = 2 \\ 0 & \text{if} \quad i = 1. \end{cases}$$

$$v^0(t) = \int_a^t [-p(s)\,x(s)]\,ds,$$

$$v^1(t) = \int_a^t [-v^0(s)]\,ds,$$

and

$$r(t)\,x''(t) = v^1(t) \qquad \text{or} \qquad [r(t)\,x''(t)]'' = p(t)\,x(t).$$

The last equation is Eq. (16).

The control problem given by (23), (24), and (25) is oscillatory of degree m if $s(b) = m$. Equation (27) is oscillatory of degree m on $a \leqslant t \leqslant b$ if there exist a vector $x = [x_1,..., x_p]$ which is J orthogonal to $\beta(b)$, and the well-defined associated control problem is oscillatory of degree m. In this case the vector x is an oscillatory solution of degree m on $a \leqslant t \leqslant b$. Equation (27) is oscillatory if for any integer $M > 0$ there exists real numbers $a < b$ such that (27) is oscillatory of degree m on $a \leqslant t \leqslant b$ and $m \geqslant M$. Finally, Eq. (27) is *nonoscillatory* if it is not oscillatory.

We note as before that problems in this setting need not be normal. Theorem 1 is always satisfied while Theorem 2 holds only if solutions are unique.

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